Continuity Exercises Solutions

1) Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Use the $\epsilon - \delta$ definition of continuity to show that $f(x)$ is continuous at $c > 0$.

Solution: Note that for $x \in X$,

$$|f(x) - f(x)| = |\sqrt{x} - \sqrt{c}| - \frac{x - c}{\sqrt{x} + \sqrt{c}} \leq \frac{1}{\sqrt{c}}|x - c|$$

Therefore for any $\epsilon > 0$, let $\delta = \sqrt{c}\epsilon$.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$\frac{x + x^3 + 5x^5}{1 + x^2}$$

Prove that $f(x)$ is continuous.

Solution: Let $g(x) = x + x^3 + 5x^5$ and $h(x) = 1 + x^2$. Consider a sequence in the domain of g, $x_n \to x_0$. We have

$$\lim g(x_n) = \lim[x_n + x_n^3 + 5x_n^5] = \lim x_n + lim x_n^3 + 5 \lim x_n^5 = x_0 + x_0^3 + 5x_0^5 = g(x_0)$$

Therefore $g(x)$ is continuous. A similar argument establishes that $h(x)$ is continuous. We know that if g and h are continuous, then g/h is continuous. Therefore f is continuous.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} e^x & \text{for} \quad x \leq 0 \\ 0 & \text{for} \quad x > 0 \end{cases}$$

Show that $f(x)$ is discontinuous.

Solution: Let $u = \ln(1/2)$. Consider the open subset of the codomain, $(1/2, 2)$. The inverse image under $(1/2, 2)$ is $(u, 0]$ which is not open. Therefore f is not continuous.

4) Let f and g be continuous at x_0 in $\mathbb{R}$. Prove that $\max(f, g)$ is continuous at x_0 (Hint: first show that for any $a, b \in \mathbb{R}$, $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$).

Solution: Note that

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$$

because f and g are real-valued. We know that the sum and difference of continuous functions is continuous. Therefore $f + g$ and $f - g$ are continuous. Now note that $|\cdot|$ is a continuous function. Therefore the composition $|f - g|$ is continuous. By the same rule, $\frac{1}{2}(f + g)$ and $\frac{1}{2}|f - g|$ are continuous. Finally, because this is the sum of two continuous functions, $\max(f, g)$ is continuous at x_0.

5) Prove that if f and g are continuous at x_0, then their product fg is continuous at x_0.

Solution: Consider a sequence in the intersection of the domain of f and g that converges to x_0. Because each is continuous we have $\lim f(x_n) = f(x_0)$ and $\lim g(x_n) = g(x_0)$. Therefore

$$\lim(fg)(x_n) = (\lim f(x_n))(\lim g(x_n)) == f(x_0)g(x_0) = (fg)(x_0)$$