

Inner Product Spaces, Orthogonality, Projection

Exercises

1. Let X and Y be normed linear spaces. Let $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ be a basis for X and $\{\mathbf{y}_1, \dots, \mathbf{y}_m\}$ a basis for Y . Prove that if $x_i \perp y_j$ for all $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, then X and Y are orthogonal spaces.

Because $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a basis for X , all $\mathbf{x} \in X$ can be expressed as linear combinations of $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ and similarly for $\mathbf{y} \in Y$. Formally, we have

$$\mathbf{x} = \sum_i \alpha_i \mathbf{x}_i$$

for all $\mathbf{x} \in X$ and arbitrary α and

$$\mathbf{y} = \sum_j \beta_j \mathbf{y}_j$$

for all $\mathbf{y} \in Y$ and arbitrary β . We need to show

$$\left\langle \sum_i \alpha_i \mathbf{x}_i, \sum_j \beta_j \mathbf{y}_j \right\rangle = 0$$

By the bilinearity of the inner product, we have

$$\begin{aligned} \left\langle \sum_i \alpha_i \mathbf{x}_i, \sum_j \beta_j \mathbf{y}_j \right\rangle &= \sum_i \langle \alpha_i \mathbf{x}_i, \sum_j \beta_j \mathbf{y}_j \rangle \\ &= \sum_i \sum_j \langle \alpha_i \mathbf{x}_i, \beta_j \mathbf{y}_j \rangle \\ &= \sum_i \sum_j \alpha_i \beta_j \underbrace{\langle \mathbf{x}_i, \mathbf{y}_j \rangle}_{=0} \end{aligned}$$

■

2. Prove: If a vector α is in the null space of a set of vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, then it is orthogonal to the space spanned by $\{\mathbf{y}_1, \dots, \mathbf{y}_m\}$ where

$$\mathbf{y}_i = \{x_{1i}, \dots, x_{ni}\}$$

Let

$$X = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

store the \mathbf{x} vectors. If $\boldsymbol{\alpha} \in N(X)$, then

$$X\boldsymbol{\alpha} = \mathbf{0}$$

or

$$\boldsymbol{\alpha}^T X^T = \mathbf{0}$$

Now let

$$X^T = \begin{bmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_m \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}$$

We can represent the space spanned by these vectors with $X^T\boldsymbol{\beta}$ with $\boldsymbol{\beta}$ taking arbitrary linear combinations of the columns of X^T . We want to show

$$\boldsymbol{\alpha}^T X^T \boldsymbol{\beta} = \mathbf{0}$$

Because $\boldsymbol{\alpha}^T X^T = \mathbf{0}$, this must be the case. ■

3. Donald Trump tweeted 100 times in April, 150 times in May, and 110 times in June.¹ Let $\mathbf{b} = (100, 150, 110)$ represent the number of tweets in each month. Project \mathbf{b} onto the linear space spanned by $\mathbf{a} = (1, 1, 1)$. Interpret your result.

We have $\mathbf{p} = \hat{x}\mathbf{a}$ and $\mathbf{e} = \mathbf{b} - \mathbf{p}$. We need \mathbf{e} to be orthogonal to \mathbf{a} , or

$$\begin{aligned} \mathbf{a}^T (\mathbf{b} - \hat{x}\mathbf{a}) &= 0 \\ \mathbf{a}^T \mathbf{b} - \hat{x}\mathbf{a}^T \mathbf{a} &= 0 \\ \mathbf{a}^T \mathbf{b} &= \hat{x}\mathbf{a}^T \mathbf{a} \\ \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} &= \hat{x} \end{aligned}$$

Substituting our values, this becomes

$$\frac{\sum_i b_i}{\sum_i 1} = \frac{100 + 150 + 110}{3} = 120$$

Notice that for n months of tweeting, this is

$$\frac{1}{n} \sum_i b_i$$

or simply the mean number of tweets.

¹Disclaimer: these data are of suspect quality.