Linear Spaces Exercises

1) Let S be a basis for X so that for every $x \in X$, there exist elements $x_1, x_2, ..., x_n \in X$ and scalars $\alpha_1, \alpha_2, ..., ...\alpha_n \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

Prove that α_i is unique for all i. (Hint: use Proposition 1). **Solution**: From the definition of basis, span$(S) = X$. Therefore every element of X can be expressed as a linear combination of elements of S. We need to show that this representation is unique. For contradiction assume there is an element $a \in X$ such that

$$a = \sum_{i=1}^{n} \alpha_i x_i$$

$$a = \sum_{i=1}^{n} \beta_i x_i$$

where the set of α_i is not the same as the set of β_i. Subtracting these yields

$$\sum_{i=1^n} (\alpha_i - \beta_i) x_i = 0$$

Because $\alpha_i \neq \beta_i$ for at least one i, the equation implies that S is not linearly independent. This contradicts the fact that S is a basis. Therefore any element in X is uniquely represented by a linear combination of the basis elements.

2) Prove or disprove the following statement: any vector space X has a unique basis. **So-**

lution: False. Trivial proof by counterexample.

3) Prove that if X is an n-dimensional linear space, then any set $S \subset X$ of $n+1$ elements is linearly dependent. (Hint: use Propositions 1 and 5).

Solution: Let $x_1, x_2, ..., x_{n+1}$ be any set of elements in X. Assume that the first n elements are linearly dependent. From (ii) this implies that there are scalars $\alpha_i \in \mathbb{R}$ such that not all are zero and

$$\sum_{i=1}^{n} \alpha_i x_i = 0$$

Therefore

$$(\sum_{i=1}^{n} \alpha_i x_i) + 0 \cdot x_{n+1} = 0$$

Therefore $x_1, ..., x_{n+1}$ must also be linearly dependent.

Now assume that $x_1, ..., x_n$ are linearly independent. We know from (i) that the n elements form a basis for X. Therefore x_{n+1} can be expressed as a linear combination of the n elements because they span X. This implies that there exist α_i such that

$$x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i$$

Subtracting x_{n+1} from both sides yields

$$(-1)x_{n+1} + \sum_{i=1}^{n} \alpha_i x_i = 0$$

By (ii) this implies that $x_1, ..., x_{n+1}$ are linearly dependent.