Logic and Proofs Exercises

1) Prove that an integer is even if and only if its square is even.

Solution: Let x be even so that $x = 2a$ for some integer a. Then $x^2 = (2a)^2 = 4a^2 = 2(2a^2)$ which is even. Therefore if x is even, x^2 is even. Now let y be an integer such that y^2 is even. Now $y^2 = 2b$ for some integer b. Therefore $y^2/2 = b$. Note that for any two numbers c and d, if $cd/2 = b$, then either $c/2$ or $d/2$ is an integer. Therefore $y/2$ must be an integer which implies that y is even. Therefore if y^2 is even, y is even. ∎

2) Let A, B, and C be any sets. Prove that

$$A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$$

(Hint: show equality by proving that both sides of the equality are subsets of each other)

Solution: First prove ($\subset$). Suppose $x \in A \backslash (B \cup C)$. Then $x \in A$ and $x \notin (B \cup C)$. Thus $x \in A$ and $[x \notin B \wedge x \notin C]$. This implies $x \in A \backslash B$ and $x \in A \backslash C$. But this is just $x \in (A \backslash B) \cap (A \backslash C)$. Now prove $\supset$. Suppose $x \in (A \backslash B) \cap (A \backslash C)$. Then $x \in (A \backslash B)$ and $x \in (A \backslash C)$. Thus $x \in A$ and $[x \notin B \wedge x \notin C]$. This implies $x \in A$ and $x \notin (B \cup C)$. But this is just $x \in A \backslash (B \cup C)$.

3) Every March in the United States, college basketball teams compete in a 64-team single-elimination tournament to determine the national champion. Assume for simplicity that rather than 64 teams, there are only 8 teams in the tournament. Let X denote the set of all teams in the tournament. Matchups in the first round are determined as follows. Each team is assigned a number 1 though 8. Refer to such a permutation as a *seeding* and denote an

individual seeding as s. Let S represent the set of all seedings, i.e. the set of all ways that the 8 teams can be assigned a number 1 through 8. In the first round the team assigned number 1 plays team number 8, team 2 plays number 7, team 3 plays team number 6, and team 4 plays team number 5. In the second round, matchups are determined similarly. The lowest seeded team plays the highest seeded team and the second highest seed plays the second lowest seed. The two remaining teams after round two play in a championship game to decide the tournament. A *tournament winner* is a team that wins three games in a row and thus wins the championship. Note that for a given seeding, the tournament winner is unique.

Assume that the result of any individual matchup is deterministic. That is, for any x and x' in X, either $x \succ x'$ or $x' \succ x$ where $\succ$ connotes "defeats." Assume that $\succ$ is exogenous i.e. it is predetermined which team beats another for all matchups (there is no randomness).

Define a *Condorcet winner* as a team $x \in X$ such that for all $x' \neq x$, $x \succ x'$.

Prove or disprove the following three statements:

i) If x is a Condorcet winner, then for all seedings $s \in S$, x is the tournament winner.

ii) If x is a tournament winner for some $s \in S$, then x is a Condorcet winner.

iii) If x is a tournament winner for all $s \in S$, then x is a Condorcet winner.

Solution

i) True. Let x be a Condorcet winner and assume FSOC that there is some s such that x is not the tournament winner. This implies that there exists some x' that x faces in s that defeats x i.e. $x' \succ x$. But by the definition of Condorcet winner, $x \succ x'$. Therefore our assumption that there exists an s such that x is not the tournament winner is false.

ii) False. Let $x' \succ x$, $x'' \succ x'$, $x \succ x''$, and $x \succ \tilde{x}$ for all other $\tilde{x} \in X$. Consider s such that x'' is seeded 2 and x' is seeded 7. In the first round x'' defeats x' and x defeats its opponent. Because x' is eliminated after round one, x defeats its opponent in rounds 2 and 3 and therefore is a tournament winner. However, because $x' \succ x$, by definition x is not a Condorcet winner.

iii) True. We will prove this by contraposition. Assume x is not a Condorcet winner. This implies that there exists some x' such that $x' \succ x$. Now consider a seeding such that x plays x' in the first round (e.g. each seeded 1 and 8 respectively). In this round x' defeats x which implies that x cannot be the tournament winner for s which means that there exists an $s \in S$ such that x is not the tournament winner. This establishes the proposition as true.