

# Normed Linear Spaces Exercises

1) Prove the following: for any  $x, y$  in a normed linear space,

$$\|x\| - \|y\| \leq \|x - y\|$$

**Solution:**  $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$  by property (4). Rearranging yields  $\|x\| - \|y\| \leq \|x - y\|$  ■

2) Prove that if  $x_n \rightarrow x$  is a convergent sequence in a normed linear space, then  $\|x_n\| \rightarrow \|x\|$

**Solution:** We are given that for all  $\epsilon > 0$ , there exists an  $N$  such that for all  $n > N$ ,  $\|x_n - x\| < \epsilon$ . The sequence  $\|x_n\|$  lies in the metric space  $(\mathbb{R}, d)$ . Therefore to show that  $\|x_n\| \rightarrow \|x\|$ , we need to show that for all  $\epsilon > 0$ , there exists an  $N$  such that for all  $n > N$ ,  $d(\|x_n\|, \|x\|) = \left| \|x_n\| - \|x\| \right| < \epsilon$ . Note that  $\left| \|x_n\| - \|x\| \right| \leq \|x_n - x\|$ : for  $a, b \in \mathbb{R}$ ,  $\|y - x\| = \max\{(y - x), (x - y)\}$ . By the reverse triangle inequality,  $\|x_n\| - \|x\| \leq \|x_n - x\|$  and  $\|x\| - \|x_n\| \leq \|x - x_n\|$ . Since  $\|x_n - x\| = \|x - x_n\|$ , we have that  $\left| \|x_n\| - \|x\| \right| \leq \|x_n - x\| < \epsilon$  for  $N$  sufficiently high. ■

3) Prove that  $\sum_{n=0}^{\infty} a\delta^n = \frac{a}{1-\delta}$  for  $\delta \in (0, 1)$ .

**Solution:** The partial sums are given by

$$s_n = \sum_{k=0}^n a\delta^k = a \frac{1 - \delta^{n+1}}{1 - \delta} :$$

$$(1 - \delta) \sum_{n=0}^{\infty} a\delta^n = \frac{a}{1 - \delta} = \sum_{k=0}^n a\delta^k - \sum_{k=0}^n a\delta^{k+1}$$

$$= a + a\delta + a\delta^2 + \dots + a\delta^n - \delta(a + a\delta + a\delta^2 + \dots + a\delta^n)$$

$$= a - a\delta^{n+1}$$

For  $\delta \in (0, 1)$ ,  $\lim_{n \rightarrow \infty} a\delta^{n+1} = 0$ . Therefore  $\lim_{n \rightarrow \infty} s_n = \frac{a}{1 - \delta}$ . ■