# Midterm Examination

POL 500 - Introduction to Mathematics for Political Science

September 10, 2019

This is a closed book examination. Calculators are not permitted. Attempt to answer all questions. Each question is worth ten points and should take about ten minutes. The exam ends sharply at 11:50.

## Question 1

Evaluate the following limits:

a.

$$\lim_{x \to 5} \frac{\sqrt{x^2 - 9} - 4}{x - 5}$$

**Solution**: Reduces to $\frac{x+5}{\sqrt{x^2-9}+4}$ where we can plug in $x = 5$ to get 10/8.

b.

$$\lim_{x \to -\infty} \frac{\sqrt{4x^6 + 9}}{2x^3 + 6x + 1}$$

**Solution**: Note that $\sqrt{4x^6}$ simplifies to $-2x^3$ on this side of zero. The standard method will then show that the limit is $-1$.

c.

$$\lim_{x \to 0} \frac{|x|}{x}$$

**Solution**: Limit DNE.

## Question 2

Let $g(x) = \ln(x)$. Use the limit definition of the derivative to find $g'(x)$. **Hint:** $\lim_{k\to 0} -(1+\frac{k}{x})^{1/k} = -e^{1/x}$

**Solution**:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0} \frac{1}{h} \ln(\frac{x+h}{x}) = \lim_{h\to 0} \ln((1+\frac{h}{x})^{1/h})$$

Note that $\lim_{h\to 0}(1+\frac{h}{x})^{1/h} = e^{1/x}$ so

$$\lim_{h\to 0} \ln((1+\frac{h}{x})^{1/h}) = \ln(e^{1/x}) = \frac{1}{x}\ln(e) = \frac{1}{x}$$

## Question 3

Compute the following derivatives with respect to $x$:

a) $f(x) = \sqrt{x^3 - 7x}$

**Solution**: $\frac{3x^2-7}{2\sqrt{x^3-7x}}$.

b) $f(x) = x^2 e^{-x}$

**Solution**: $x(2-x)e^{-x}$

c) $f(x) = x^2\sqrt{1-x^2}$

**Solution**: $2x\sqrt{1-x^2} - \frac{x^3}{\sqrt{1-x^2}}$

## Question 4

Let $f(x) = x^2$ and $g(x) = x$. Let $A = \{(x,y) \in \mathbb{R}^2 : y \geq f(x)\}$ and $B = \{(x,y) \in \mathbb{R}^2 : y \leq g(x)\}$.

a) $A \cap B$ is a region in $\mathbb{R}^2$. Find the area of this region.

**Solution**: $f(x)$ and $g(x)$ intersect at 0 and 1. The area is therefore given by

$$\int_0^1 (x - x^2)dx = \int_0^1 x dx - \int_0^1 x^2 dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

square units.

b) Now assume that a point is drawn completely at random from $A \cap B$. Find the joint probability density function that represents this process.

**Solution**: $f(x, y) = 6$ if $(x, y) \in A \cap B$, 0 otherwise.

## Question 5

Evaluate the following integrals

a) $\int_0^2 (x^2 - xb)dx$

**Solution**: $-\frac{6b-8}{3}$

b) $\int x \ln(x)dx$

**Solution**: $\int x \ln(x)dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2}\frac{1}{x}dx = \frac{x^2}{2} \ln(x) - (1/2) \int xdx = \frac{x^2}{2} \ln(x) - (1/2)\frac{x^2}{2} + C$

## Question 6

A family has two children. Given that one of the children is a boy and that he was born on a Tuesday, what is the probability that both children are boys?

**Solution**: Let $B$ be the event that the family has one boy born on Tuesday and $A$ be the event that both children are boys. Note that there are 49 permutations for days of the week the boys were born on and 13 of these have a boy born on a Tuesday so $Pr(B|A) = \frac{13}{49}$. $Pr(A) = \frac{1}{4}$. There are $14^2 = 196$ ways to select the gender and day of the week the child was born on. Of these, $13^2 = 169$ ways do not have a boy born on Tuesday so $P(B) = \frac{27}{196}$. By Bayes' rule then $Pr(A|B) = \frac{13}{27}$.

## Question 7

Prove the following statement: If $A$ is orthogonal, then the rows of $A$ are orthogonal to each other and each row has a norm of 1. **Hint: A matrix is orthogonal if $A^T A = I$**

**Solution**: Let $n$ denote the number of rows and $\boldsymbol{a}_i$ denote the $i$th row of $A$. Note that $\boldsymbol{a}_i$ is a row vector, not a column vector. So its dot product is written as $\boldsymbol{a}_i\boldsymbol{a}_i^\top$.

$$
\begin{aligned}
AA^\top &= \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1^\top & \dots & \boldsymbol{a}_n^\top \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{a}_1\boldsymbol{a}_1^\top & \boldsymbol{a}_1\boldsymbol{a}_2^\top & \dots & \boldsymbol{a}_1\boldsymbol{a}_n^\top \\ \boldsymbol{a}_2\boldsymbol{a}_1^\top & \boldsymbol{a}_2\boldsymbol{a}_2^\top & \dots & \boldsymbol{a}_2\boldsymbol{a}_n^\top \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{a}_n\boldsymbol{a}_1^\top & \boldsymbol{a}_n\boldsymbol{a}_2^\top & \dots & \boldsymbol{a}_n\boldsymbol{a}_n^\top \end{bmatrix} \\
&= I
\end{aligned}
$$

The last equality implies that all diagonal elements, $\boldsymbol{a}_i\boldsymbol{a}_i^\top = \|\boldsymbol{a}_i\|^2, i = 1, \dots, n$, should be equal to 1. In addition, all off-diagonal elements, $\boldsymbol{a}_i\boldsymbol{a}_j^\top = \boldsymbol{a}_i \cdot \boldsymbol{a}_j, i \neq j$, should be equal to 0. Therefore, each row has a norm of 1 and the rows are orthogonal to each other.

## Question 8

Use the Cauchy-Schwartz Inequality

$$\boldsymbol{u} \cdot \boldsymbol{v} \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

to prove the Triangle Inequality

$$\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$$

**Hint: Convince yourself that** $\|\boldsymbol{u} + \boldsymbol{v}\|^2 = (\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v})$

**Solution:**

$$\begin{aligned}
\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) \\
&= \boldsymbol{u}\cdot\boldsymbol{u} + 2\boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{v} \\
&= \|\boldsymbol{u}\|^2 + 2\boldsymbol{u}\cdot\boldsymbol{v} + \|\boldsymbol{v}\|^2 \\
&\leq \|\boldsymbol{u}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^2 &= (\|\boldsymbol{u}\|+\|\boldsymbol{v}\|)^2
\end{aligned}$$

And because norms are strictly positive we can conclude

$$\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$$

as desired.