Practice Midterm Examination

POL 500 - Introduction to Mathematics for Political Science

August 17, 2018

Question 1

Evaluate the following limits:

a)

$$\lim_{x \to 3} \frac{x^3 - 27}{x^4 - 5x^3 + 6x^2}$$

Solution:

$$\lim_{x \to 3} \frac{x^3 - 27}{x^4 - 5x^3 + 6x^2} = \lim_{x \to 3} \frac{(x-3)(x^2+3x+9)}{x^2(x-3)(x-2)} = \lim_{x \to 3} \frac{x^2+3x+9}{x^2(x-2)} = 3$$

b)

$$\lim_{x \to \infty} \frac{(x^4 + 3x - 99)(2 - x^5)}{(18x^7 + 9x^6 - 3x^2 - 1)(x+1)}$$

Solution:

$$\lim_{x \to \infty} \frac{(x^4 + 3x - 99)(2 - x^5)}{(18x^7 + 9x^6 - 3x^2 - 1)(x+1)} = \lim_{x \to \infty} \frac{(1 + 3/x^3 - 99/x^2)(-2/x^5 + 1)}{(1 + 9/18x - 3/18x^5 - 1/18^7)(1 + 1/x)} \frac{x^4(-x^5)}{(18x^7)x}$$

$$= \lim_{x \to \infty} \frac{-x}{18} = -\infty$$

c)

$$\lim_{x \to \infty} \frac{-5x^5}{\sqrt[3]{27x^6 + 8x}(\sqrt{4x^6 - 5x^5} + 2x^3)}$$

Solution:

$$\lim_{x\to\infty} \frac{-5x^5}{\sqrt[3]{27x^6+8x}(\sqrt{4x^6-5x^5}+2x^3)} = \lim_{x\to\infty} \frac{-5x^5}{(\frac{\sqrt[3]{27x^6+8x}}{\sqrt[3]{27x^6}} \times (3x^2))(\frac{\sqrt{4x^6-5x^5}+2x^3}{4x^3} \times (4x^3))}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{\sqrt[3]{27x^6+8x}}{\sqrt[3]{27x^6}} \frac{\sqrt{4x^6-5x^5}+2x^3}{4x^3}} \frac{-5x^5}{(3x^2)(4x^3)} = \frac{-5}{12}$$

Questions 2

Use the limit definition of the derivative to evaluate the derivative of $f(x) = \frac{1}{x+3}$.

Solution: $f'(x) = \lim_{h\to 0} \frac{1/(x+h+3)-1(x+3)}{h} = \frac{-h/(x+3)(x+h+3)}{h} = \frac{-1}{(x+3)^2}$

Question 3

Compute the following derivatives with respect to x:

a) $f(x) = \frac{3x^2+1}{2x^8-7}$

Solution: Use the quotient rule to get $f'(x) = \frac{(2x^8-7)(6x)-(3x^2+1)(16x^7)}{(2x^8-7)^2}$

b) $f(x) = (\frac{\sqrt{x}}{1+x})^2$

Solution: $f'(x) = \frac{1-x}{(x+1)^3}$

c) $f(x) = \frac{\ln(3x+2)}{3x+2}$

Solution: $f'(x) = \frac{3(1-\ln(3x+2))}{(3x+2)^2}$

Question 4

Evaluate the following integrals:

a) $\int x\sqrt[5]{3x+2}dx$

Solution: Let $t = \sqrt[5]{3x+2}$. Now $t^5 = 3x+2$ and $5t^4dt = 3dx$ or $dx = \frac{5}{3}t^4dt$. This yields $\int x\sqrt[5]{3x+2}dx = \int \frac{1}{3}(t^5-2)(t) \times \frac{5}{3}t^4dt = \frac{5}{9}\int(t^{10}-2t^5)dt = \frac{5}{99}t^{11} - \frac{5}{27}t^6 + C$.

b) $\int \frac{\ln(x)}{\sqrt{(x)}}dx$

Solution: Let $f = \ln(x)$, $g' = \frac{1}{\sqrt{x}}$ and integrate by parts i.e. $\int fg' = fg - \int f'g$. $f' = \frac{1}{x}, g = 2\sqrt{x}$. $2\sqrt{x}\ln(x) - \int \frac{2}{\sqrt{x}}dx = 2\sqrt{x}\ln(x) - 4\sqrt{x}$

Question 5

Three different machines, M_1, M_2, and M_3 were used for producing a large batch of similar manufactured items. Suppose that 20 percent of the items were produced by M_1, 30 percent by M_2, and 50 percent by M_3. Suppose further that 1 percent of items produced by M_1 are defective, 2 percent produced by M_2 are defective, and 3 percent produced by M_3 are defective. Finally suppose that one item is selected at random from the batch and it is found to be defective. What is the probability that this item was produced by M_2?

Solution: Let $Pr(B_i)$ be the probability that the item selected was produced by M_i. By Bayes' rule,

$$Pr(B_2|A) = \frac{Pr(B_2)Pr(A|B_2)}{\sum_{j=1}^{3} Pr(B_j)Pr(A|B_j)} = \frac{(.3)(.02)}{(.2)(.01) + (.3)(.02) + (.5)(.03)} = .26$$

Question 6

Suppose that the p.d.f. of a random variable X has the following form:

$$f(x) \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{1}{2} & \text{for} \quad x \in [0,1] \\ cx & \text{for} \quad x \in [1,2] \\ 0 & \text{for} \quad x > 2 \end{cases}$$

Find c.

Solution: We need $\int_1^2 csdx = \frac{1}{2}$. $c = \frac{1}{3}$ accomplishes this.

Question 7

Use Gauss-Jordan reduction to solve the following system of equations:

$2y + 3z = 7$

$x + y - z = -2$

$-x + y - 5z = 0$

Solution: $(-3, 2, 1)$.

Question 8

The angle between the vectors $(1, 0, -1, 3)$ and $(1, \sqrt{3}, 3, -3)$ in $\mathbb{R}^4$ is $a\pi$. Find a.

Solution: Recall that the angle between two vectors is given by $\arccos \frac{x \cdot y}{\|x\|\|y\|}$. This gives us $a\pi = \arccos(\frac{-11}{\sqrt{11}\sqrt{22}}) = \arccos(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$.