# IMPS 2019: Final Exam

**Instructions**: This is a closed book examination. Calculators are not permitted. **There are 8 questions, from which you can choose 6 to answer.** Each question is worth ten points and should take about 30 minutes to complete. You have three hours.

1. Let $S$ and $T$ be sets. Show

$$(S \cap T)^c = S^c \cup T^c$$

Take $x \in (S \cap T)^c \implies x \notin S \cap T \implies x \notin S \vee x \notin T \implies x \in S^c \vee x \in T^c \implies x \in S^c \cup T^c$. So $(S \cap T)^c \subseteq S^c \cup T^c$. Now take $x \in S^c \cup T^c \implies x \in S^c \vee x \in T^c \implies x \notin S \wedge x \notin T \implies x \notin S \cap T \implies x \in (S \cap T)^c$. So $S^c \cup T^c \subseteq (S \cap T)^c$. If $(S \cap T)^c \subseteq S^c \cup T^c$ and $S^c \cup T^c \subseteq (S \cap T)^c$ then $(S \cap T)^c = S^c \cup T^c$. ■

2. Consider a sequence $\{x_n\} \in \left(\mathbb{R}^2, d_2\right)$ with

$$x_n = \left(\frac{1}{n}, -\frac{1}{n}\right)$$

for all $n$. Does this sequence converge? Prove your answer.

**Solution** The sequence converges to $(0, 0)$. To see why, let

$$N^\star = \lceil \frac{\sqrt{2}}{\epsilon} \rceil$$

Then, for all $n > N^\star$

$$\begin{aligned} d_2(x_n, 0) &= \sqrt{\left(\frac{1}{n}\right)^2 + \left(-\frac{1}{n}\right)^2} \\ &< \sqrt{\left(\frac{1}{N^\star}\right)^2 + \left(-\frac{1}{N^\star}\right)^2} \\ &\leq \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(-\frac{\epsilon}{\sqrt{2}}\right)^2} \\ &= \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} \\ &= \epsilon \end{aligned}$$

■

3. Let $f, g$ be continuous functionals on a metric space $(X, d)$. Prove $f + g$ is continuous.

**Solution**

If $f$ and $g$ are continuous, then for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \frac{\epsilon}{2}$$

$$|x_1 - x_2| < \delta \implies |g(x_1) - g(x_2)| < \frac{\epsilon}{2}$$

Then

$$|f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| < \epsilon$$

By the triangle inequality we have

$$|f(x_1) - f(x_2) + (g(x_1) - g(x_2))| \leq |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| < \epsilon$$
$$|(f(x_1) + g(x_1)) - (f(x_2) + g(x_2))| \leq |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| < \epsilon$$

which shows $f + g$ is continuous. ■

4. Use the definition of a strictly convex function to show that if $f(x) = x^2$ then $f(x)$ is strictly convex. Proofs employing derivatives will not be accepted.

**Solution**

The definition of strict concavity requires that for $x \neq y$ and $\lambda \in (0, 1)$ we have $\lambda f(x) + (1 - \lambda)f(y) > f(\lambda x + (1 - \lambda)y)$. The latter would require that:

$$\lambda x^2 + (1 - \lambda)y^2 - (\lambda x + (1 - \lambda)y)^2 > 0$$

But

$$\lambda x^2 + (1 - \lambda)y^2 - (\lambda x + (1 - \lambda)y)^2 = \lambda x^2 + (1 - \lambda)y^2 - \lambda^2 x^2 - 2\lambda(1 - \lambda)xy - (1 - \lambda)^2 y^2$$
$$= \lambda(1 - \lambda)x^2 + \lambda(1 - \lambda)y^2 - 2\lambda(1 - \lambda)xy$$

Thus, we need to show that

$$\lambda(1 - \lambda)x^2 + \lambda(1 - \lambda)y^2 - 2\lambda(1 - \lambda)xy > 0$$

Since $\lambda$ and $(1 - \lambda)$ are both larger than zero we can divide both sides by $\lambda(1 - \lambda)$ which implies that

$$x^2 - 2xy + y^2 > 0$$

or

$$(x - y)^2 > 0$$

which holds for all $x, y$. ■

5. Let $X$ be an innner product space. Show that if

$$\boldsymbol{x}_1^T \boldsymbol{x}_2 \leq \|\boldsymbol{x}_1\| \|\boldsymbol{x}_2\|$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in X$, then

$$\|\boldsymbol{x}_1 + \boldsymbol{x}_2\| \leq \|\boldsymbol{x}_1\| + \|\boldsymbol{x}_2\|$$

**Solution** Recall that in a normed linear space

$$\|\boldsymbol{x}\|^2 = \boldsymbol{x}^T \boldsymbol{x}$$

We therefore know

$$\begin{aligned}
(\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|)^2 &= (\boldsymbol{x}_1 + \boldsymbol{x}_2)^T (\boldsymbol{x}_1 + \boldsymbol{x}_2) \\
&= \boldsymbol{x}_1^T \boldsymbol{x}_1 + 2\boldsymbol{x}_1^T \boldsymbol{x}_2 + \boldsymbol{x}_2^T \boldsymbol{x}_2 \\
&= \|\boldsymbol{x}_1\|^2 + 2\boldsymbol{x}_1^T \boldsymbol{x}_2 + \|\boldsymbol{x}_2\|^2
\end{aligned}$$

by the bilinearity of inner products. Applying our premise, we must have

$$\begin{aligned}
\|\boldsymbol{x}_1\|^2 + 2\boldsymbol{x}_1^T \boldsymbol{x}_2 + \|\boldsymbol{x}_2\|^2 &\leq \|\boldsymbol{x}_1\|^2 + 2\|\boldsymbol{x}_1\|\|\boldsymbol{x}_2\| + \|\boldsymbol{x}_2\|^2 \\
(\|\boldsymbol{x}_1 + \boldsymbol{x}_2\|)^2 &\leq (\|\boldsymbol{x}_1\| + \|\boldsymbol{x}_2\|)^2 \\
\|\boldsymbol{x}_1 + \boldsymbol{x}_2\| &\leq \|\boldsymbol{x}_1\| + \|\boldsymbol{x}_2\|
\end{aligned}$$

as desired. ■

6. Use the definition of a differentiable function to show that if $f, g : X \to Y$ are differentiable at $\boldsymbol{x}_0$ then $f + g$ is also differentiable with derivative

$$D(f+g)[\boldsymbol{x}_0] = Df[\boldsymbol{x}_0] + Dg[\boldsymbol{x}_0]$$

**Solution**

If $f$ is differentiable at $\boldsymbol{x}_0$, then there exists a linear function $h_f(\boldsymbol{x})$ such that $\boldsymbol{x} \to \boldsymbol{0} \implies \eta_f(\boldsymbol{x}) \to \boldsymbol{0}$ where

$$\eta_f(\boldsymbol{x}) = \frac{f(\boldsymbol{x}_0 + \boldsymbol{x}) - (f(\boldsymbol{x}_0) + h_f(\boldsymbol{x}))}{\|\boldsymbol{x}\|}$$

Similarly for $g$, there exists a linear function $h_g(\boldsymbol{x})$ such that $\boldsymbol{x} \to \boldsymbol{0} \implies \eta_g(\boldsymbol{x}) \to \boldsymbol{0}$ where

$$\eta_g(\boldsymbol{x}) = \frac{g(\boldsymbol{x}_0 + \boldsymbol{x}) - (g(\boldsymbol{x}_0) + h_g(\boldsymbol{x}))}{\|\boldsymbol{x}\|}$$

Combining these gives

$$\begin{aligned}
\eta_{fg}(\boldsymbol{x}) = \eta_f(\boldsymbol{x}) + \eta_g(\boldsymbol{x}) &= \frac{f(\boldsymbol{x}_0 + \boldsymbol{x}) - (f(\boldsymbol{x}_0) + h_f(\boldsymbol{x}))}{\|\boldsymbol{x}\|} + \frac{g(\boldsymbol{x}_0 + \boldsymbol{x}) - (g(\boldsymbol{x}_0) + h_g(\boldsymbol{x}))}{\|\boldsymbol{x}\|} \\
&= \frac{f(\boldsymbol{x}_0 + \boldsymbol{x}) + g(\boldsymbol{x}_0 + \boldsymbol{x}) - (f(\boldsymbol{x}) + g(\boldsymbol{x}) + h_f(\boldsymbol{x}) + h_g(\boldsymbol{x}))}{\|\boldsymbol{x}\|}
\end{aligned}$$

Since $h_f$ and $h_g$ are linear functions,

$$h_{fg}(\boldsymbol{x}) = h_f(\boldsymbol{x}) + h_g(\boldsymbol{x})$$

is also a linear function. Moreover $\boldsymbol{x} \to \mathbf{0} \implies \eta_f(\boldsymbol{x}) + \eta_g(\boldsymbol{x}) \to \mathbf{0} \implies \eta_{fg}(\boldsymbol{x}) \to 0$ which is sufficient to demonstrate the differentiability of $f + g$. ■

7. Consider the $n$ equation system

$$\begin{aligned} y_1 &= \beta x_1 + \epsilon_1 \\ &\cdots \\ y_n &= \beta x_n + \epsilon_n \end{aligned}$$

where $\beta$ is a scalar. Show that the $\hat{\beta}$ that minimizes the sum of squared errors $(\sum_{i=1}^n \epsilon_i^2)$ is given by

$$\hat{\beta} = \frac{\sum_{i=1}^n y_i x_i}{\sum_{i=1}^n x_i^2}$$

**Solution**

The sum of squared errors can be written

$$f_{\text{SSE}}(\beta) = \sum_{i=1}^n \epsilon_i^2 = \sum_{i=1}^n (y_i - \beta x_i)^2$$

Then, by the chain rule

$$\begin{aligned} \frac{\partial f_{\text{SSE}}(\beta)}{\partial \beta} &= \sum_{i=1}^n 2\,(y_i - \beta x_i)\,(-x_i) = 0 \\ \sum_{i=1}^n 2\left(-y_i x_i + \beta x_i^2\right) &= 0 \\ 2\sum_{i=1}^n \beta x_i^2 &= 2\sum_{i=1}^n y_i x_i \\ \beta \sum_{i=1}^n x_i^2 &= \sum_{i=1}^n y_i x_i \\ \hat{\beta} &= \frac{\sum_{i=1}^n y_i x_i}{\sum_{i=1}^n x_i^2} \end{aligned}$$

8. Consider the optimization problem

$$\max_x f(x;\theta) = (1-x)x + \theta(1-x)$$

Show that the function is strictly concave in $x$ and write the value function $V(\theta) = f(x^\star(\theta);\theta)$. How does $x^\star(\theta)$ change with $\theta$?

**Solution**

The first order partial of the objective function with respect to $x$ is given by

$$\frac{\partial f(x;\theta)}{\partial x} = 1 - 2x - \theta$$

Then the second order partial is

$$\frac{\partial^2 f(x;\theta)}{\partial x^2} = -2 < 0$$

so the objective function is concave in $x$. To find $x^\star(\theta)$ let

$$\frac{\partial f(x;\theta)}{\partial x} = 1 - 2x - \theta = 0 \implies x^\star(\theta) = \frac{1-\theta}{2}$$

Plugging this back into the objective function gives the value function

$$\begin{aligned}
V(\theta) = f(x^\star(\theta);\theta) &= (1 - x^\star(\theta))x^\star(\theta) + \theta(1 - x^\star(\theta)) \\
&= x^\star(\theta) - x^\star(\theta)^2 + \theta - \theta x^\star(\theta) \\
&= \left(\frac{1-\theta}{2}\right) - \left(\frac{1-\theta}{2}\right)^2 + \theta - \left(\frac{\theta(1-\theta)}{2}\right) \\
&= \left(\frac{1-\theta}{2}\right) - \left(\frac{1-\theta}{2}\right)^2 + \frac{\theta+\theta^2}{2} \\
&= \left(\frac{1+\theta^2}{2}\right) - \left(\frac{1-\theta}{2}\right)^2 \\
&= \left(\frac{2+2\theta^2}{4}\right) - \frac{(1-\theta)^2}{4} \\
&= \frac{\theta^2+2\theta+1}{4} \\
&= \frac{1}{4}(1+\theta)^2
\end{aligned}$$

To consider how $x^\star(\theta)$ changes with $\theta$, we simply take the function's derivative with respect to $\theta$.

$$\frac{\partial x^\star(\theta)}{\partial \theta} = -\frac{1}{2}$$